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TOPOLOGICAL GROUPS WITH THIN GENERATING SETS

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TOPOLOGICAL GROUPS WITH THIN GENERATING SETS

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Abstract. A discrete subset S of a topological group G with identity 1 is called suitable for G if S generates a dense subgroup of G and $S\cup\{1\}$ is closed in G. We study various algebraic and topological conditions on a group G which imply the existence of a suitable set for G as well as the restraints imposed by the existence of such a set. The classes $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ of topological groups having a closed, generating and a closed generating suitable set are considered. The problem of stability of these classes under the product, direct sum operations and taking subgroups or quotients is investigated. We show that (totally) minimal Abelian groups often have a suitable set. It is also proved that every Abelian group endowed with the finest totally bounded group topology has a closed generating suitable set. More generally, the Bohr topology of every locally compact Abelian group admits a suitable set.

Keywords: Suitable set, convergent sequence, torsion, divisible, monothetic, (totally) minimal, quotient group, Bohr topology, free topological group, countably compact, pseudocompact, separable, Lindelöf, ω-bounded group

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0. Introduction. It is a typical approach in the realm of topological groups to impose some kind of finiteness condition on a group which involves somehow the topology of the group. The simplest one is to ask the group to have a dense finitely generated subgroup. Let us denote by $\mathcal{F}$ the class of these (*topologically finitely generated*) groups. It is clear that all groups in $\mathcal{F}$ are separable. The groups in $\mathcal{F}$ that have a dense cyclic subgroup are known also as *monothetic* groups, they are necessarily Abelian. It is well known that a compact connected Abelian group G is monothetic iff $w(G) \leq \mathfrak{c}$ (cf. [HR]). The following extension of this result was obtained in [HM] (see also [CM, Theorem 3.2] for a recent advance in the locally compact case):

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0.1. Theorem. *A compact connected group G has a dense 2-generated subgroup iff $w(G) \leq \mathfrak{c}$.*

Out of the class of connected groups the condition to be topologically finitely generated turns out to be rather restrictive (even for compact Abelian groups) as the following examples show. In particular, (c) explains the rôle of "connected" in Theorem 0.1.

(a) The group $\mathbb{Q}$ of rationals equipped with the usual interval topology does not belong to $\mathcal{F}$ (for every finitely generated subgroup of $\mathbb{Q}$ is closed and discrete).

(b) A dense subgroup of a monothetic metrizable group need not be topologically finitely generated (a standard example is the torsion subgroup of the circle group $\mathbb{T}$).

(c) A compact totally disconnected group in $\mathcal{F}$ is metrizable (see Proposition 2.1).

The above examples (a)–(c) suggest the replacement of the rigid condition of *finiteness* of the set S of topological generators of a group G by an appropriate weaker one. It is easy to see that every dense subgroup G of a monothetic metrizable group K has a dense subgroup generated by a closed discrete subset. Indeed, take a sequence $\{x_n : n \in \omega\}$ in G that converges to a topological generator a of K that does not belong to G (if such an a is not available, then G itself is monothetic). The set $S = \{x_n : n \in \omega\}$ is as required. Obviously, this can be done for a topologically finitely generated group K as well. This simple fact suggests to consider the following two possible substitutes of finiteness of S. The first of them was proposed by Tate (cf. [Do]) and studied later by Douady [Do], Mel'nikov [M] and Hofmann and Morris [HM], the second (as well as the first one) was considered recently in [CMRST] (which will be abbreviated as [C–T]):

(i) S is discrete and $\overline{S} \subseteq S \cup \{1\}$;

(ii) S is closed and discrete in G.

Denote by $\mathcal{S}$ (resp. $\mathcal{S}_c$) the class of groups G having a subset S as in (i) (resp. (ii)) with $\overline{\langle S \rangle} = G$. It turns out that very often the subset S of the group G has the stronger property to *generate* G, instead of generating just a dense subgroup of G. We denote by $\mathcal{S}_g$ and $\mathcal{S}_{cg}$, respectively, the corresponding subclasses of $\mathcal{S}$ and $\mathcal{S}_c$. A subset S of G as in (i) with $\overline{\langle S \rangle} = G$ will be called *suitable*.

It is not surprising that the classes of compact and locally compact groups groups were the first to be considered on the subject to have a suitable set. The following fundamental result is due to Hofmann and Morris (see Theorem 1.12 of [HM]):

0.2. Theorem. *The class $\mathcal{S}$ contains all locally compact groups.*

For totally disconnected compact groups this was proved earlier by Tate, cited in [Do].

The existence of several wide classes of non locally compact topological groups with suitable sets was recently established in [C–T]. In terms of the classes $\mathcal{S}$, $\mathcal{S}_g$, $\mathcal{S}_c$ and $\mathcal{S}_{cg}$ we can summarize the results of [C–T] as follows.

0.3. Theorem. *The class $\mathcal{S}_c$ contains the following subclasses:*
(a) all metrizable non-compact groups;
(b) all groups G with $d(G) < b(G)$, i.e., $b(G) = d(G)^+$;
(c) the free (Abelian) topological groups on separable Tikhonov spaces;
(d) non-compact separable groups of countable pseudocharacter.

Items (a), (b) and (c) of the above theorem follow respectively from Theorems 6.6, 5.7 and 5.1 of [C–T]. In its turn, item (d) follows from Theorems 5.14 and 6.3 of [C–T]. The definition of $b(G)$ is given in Section 1. We just note here that $b(G) \leq \aleph_0$ iff G is precompact, and $b(G) \leq \aleph_1$ iff G is $\aleph_0$-bounded in the sense of Guran [Gu].

The study of the class $\mathcal{S}_{cg}$ is very far from being complete. The following result is Theorem 2.2 of [C–T].

0.4. Theorem. *$\mathcal{S}_{cg}$ contains all countable topological groups.*

According to Theorem 6.3 of [C–T], topological groups of countable pseudocharacter have special features:

0.5. Theorem. *For a non-compact group G of countable pseudocharacter, $G \in \mathcal{S}$ is equivalent to $G \in \mathcal{S}_c$ and $G \in \mathcal{S}_g$ is equivalent to $G \in \mathcal{S}_{cg}$.*

By Corollary 3.10 of [C–T], the free Abelian topological group on $\beta\omega_1$ does not have a suitable set. Since the free Abelian topological group on a compact space is complete [Gr, Theorem 6], we obtain the following result:

0.6. Theorem. *There exists a σ-compact complete Abelian group $G \notin \mathcal{S}$.*

Note that every separable σ-compact group has a suitable set by Corollary 3.9 of [DTT], so that the group G in the above theorem is necessarily non-separable.

Another generalization of compactness is countable compactness. By Theorem 3.15 of [C–T], under MA there exists a separable countably compact topological group with no suitable set. The group in question is exactly the one constructed by van Douwen [vD1] to show that countable compactness is not productive in the class of topological groups. The basic properties of van Douwen's group G are the following: G is infinite, of order 2 and it does not contain non-trivial convergent sequences. This gave rise to the problem as to whether there exists a ZFC example of a pseudocompact topological group without suitable set (see Open Question 3 of [C–T]). The positive answer to the problem was recently obtained in [DTT] in the following strong form:

0.7. Theorem. *There exists a countably compact topological group $G \notin \mathcal{S}$ such that every countable subset of G is contained in a compact subgroup of G.*

In Section 2 we isolate several necessary conditions on a group G, primarily of topological nature, for $G \in \mathcal{S}$ or $G \in \mathcal{S}_c$, etc. The results of Section 2 will clarify the difference between compact, countably compact and pseudocompact groups when suitable sets are involved.

In Section 3 we study the behaviour of the classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ under the basic operations: cartesian products, direct sums, passing to subgroups and taking quotients.

The existence of a suitable set for minimal and totally minimal groups is considered in Section 4. We show that all totally minimal Abelian groups as well as totally minimal connected precompact groups belong to $\mathcal{S}$ (see Theorems 4.1.4 and 4.1.5). In Theorem 4.2.1 we establish that every minimal countably compact connected Abelian group has a suitable set. Making use of the example found in [DTT] relevant to Theorem 0.7 we produce in Example 4.2.3 ω-bounded (hence countably compact) minimal Abelian groups that have no suitable set.

In Section 5 we study locally compact Abelian groups G endowed with their Bohr topology G^+. We prove that for every locally compact Abelian group G the group G^+ has a suitable set (Theorem 5.8). Moreover, for every Abelian group G, the topological group $G^\# \in \mathcal{S}_{cg}$ (i.e., G equipped with the maximal totally bounded group topology has a closed generating suitable set, Theorem 5.7).

This paper is a logical continuation of a more topological investigation of groups in $\mathcal{S}$ and $\mathcal{S}_c$ done by the same group of authors [DTT]. Therefore, several results presented here depend on the content of [DTT]. For the reader's convenience we tried to reduce to minimum such crossreferences and supplied the paper with almost all necessary material.

1. Notation and Terminology

We denote by $\mathbb{R}$, $\mathbb{I}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively the reals, the unit interval $[0,1]$, the rationals, the integers and natural numbers. The circle group $\mathbb{R}/\mathbb{Z}$ is denoted by $\mathbb{T}$. The groups $\mathbb{R}$ and $\mathbb{T}$ are assumed to carry their usual additive group operations and topology. We also use $\mathbf{J}_p$ to denote the (compact) group of p-adic integers.

Let G be a group. The neutral element of G is denoted by 1 or 1_G in general, or respectively by 0 or 0_G if G is Abelian. The minimal subgroup of G containing a subset $A \subseteq G$ is $\langle A \rangle$. The group G is *divisible* if for every $g \in G$ and $n \in \mathbb{N}^+$ the equation $x^n = g$ has a solution in G.

Topological groups are assumed to be Hausdorff. We denote by $\widehat{G}$ the two-sided (Raĭkov) completion of G and by $c(G)$ the connected component of 1 in G. A group G is *precompact* (or, equivalently, *totally bounded*) if $\widehat{G}$ is compact, *pseudocompact* if every continuous real-valued function on G is bounded, *countably compact* – if each open countable cover of G admits a finite subcover. A group G is called *ω-bounded* if every countable subset of G is contained in a compact subgroup. Every ω-bounded group is countably compact, but not vice versa.

By $b(G)$ we denote the minimal cardinal number such that the group G can be covered by less than $b(G)$ translates of every neighborhood of identity in G. Therefore, a totally bounded group G is characterized by $b(G) \le \omega$. Section 5 of [C-T] contains more details on the properties of the cardinal function b and relations between this function and the cellularity, density, etc.

A topological group (G, τ) is called *minimal* if τ is a minimal element of the partially ordered (with respect to inclusion) set of Hausdorff group topologies on the group G; G is *totally minimal* if every Hausdorff quotient of G is minimal.

The closure of a subset $Y \subseteq X$ in X is denoted by $cl_X Y$ or simply $cl\, Y$ if there is no ambiguity. When convenient, we also use $\overline{Y}^X$ or $\overline{Y}$ for the same purpose.

Throughout the paper, we will use the notions of σ- and Σ-*products* of topological groups. Let us describe them briefly. Let 1 be the identity of the cartesian product $\Pi = \prod_{\alpha \in A} G_\alpha$ of groups G_α and for every $x \in \Pi$, define $supp(x) = \{\alpha \in A : x_\alpha \neq 1_\alpha\}$. The σ-*product* and Σ-*product* of the groups G_α with the *center* a are defined respectively as follows:

$$\sigma(A) = \{x \in \Pi : |supp(x)| < \aleph_0\}$$

and

$$\Sigma(A) = \{x \in \Pi : |supp(x)| \le \aleph_0\}.$$

The subgroups $\sigma(A)$ and $\Sigma(A)$ of Π are considered with the topology inherited from Π. Both $\sigma(A)$ and $\Sigma(A)$ are dense subgroups of Π. The subgroup $\sigma(A)$ of Π is also called the *direct* (or *weak*) *sum* of the groups G_α, $\alpha \in A$.

The cardinality of the continuum 2^ω will be denoted by $\mathfrak{c}$. The notation for cardinal functions is standard: $w(X)$, $nw(X)$, $d(X)$, $\chi(X)$, $\psi(X)$ and $L(X)$ stand for the weight, network weight, density, character, pseudocharacter and Lindelöf number of X respectively.

The abbreviations CH and MA are used for the Continuum Hypothesis and Martin's Axiom respectively. The symbol $\diamondsuit$ refers to a special set-theoretic axiom concerning unbounded subsets of ω_1. It is well known that $\diamondsuit \Longrightarrow$ CH $\Longrightarrow$ MA (see [K]).

2. Some necessary conditions for $G \in \mathcal{S}$

We begin with the study of countably compact topological groups that have a (closed, generating) suitable set. The following auxiliary result describes the structure of totally disconnected countably compact groups with a finite suitable set. Let F_n be the free group of n generators and let τ_{fin} be its pro-finite topology, namely the topology that has as basic neighborhoods of 1 all normal subgroups of finite index. The completion $\widehat{F}_n$ of the group (F_n, τ_{fin}) is called the *free profinite group* of

n generators. This name is justified by the fact that $\widehat{F}_n$ has the universal property of extending continuous homomorphisms to profinite groups of n generators (see the argument in the proof of Proposition 2.1 below).

2.1. Proposition. *(a) A totally disconnected countably compact group G is topologically n-generated for some $n \in \mathbb{N}^+$ iff it is a quotient of the free profinite group $\widehat{F}_n$ of n generators. In particular, G is metrizable, hence compact.*

(b) A countably compact group $H \in \mathcal{F}$ is essentially connected: the group $H/c(H)$ is a quotient of $\widehat{F}_n$ for some $n \in \mathbb{N}^+$. This yields that $c(H)$ is a G_δ-subgroup of H and $H/c(H)$ is metrizable.

Proof. (a) Let us first consider the case of a compact group G. Note that the group (F_n, τ_{fin}) is metrizable because every finite-index subgroup of F_n is finitely generated (by the Nielsen-Schreier theorem, see Theorem 6.1.1 of [Ro]). Thus, there are only countably many such subgroups of F_n, and hence $\widehat{F}_n$ is metrizable as well. Suppose that a totally disconnected compact group G is topologically n-generated. Then G is profinite, so it is a quotient of $\widehat{F}_n$. To see this, fix any n-generated subgroup H of G and a surjective homomorphism $f\colon F_n \to H$. Then f is continuous when F_n is endowed with τ_{fin}, so that f extends to a continuous homomorphism $\hat{f}\colon \widehat{F}_n \to G$ which has to be surjective as $\hat{f}(\widehat{F}_n)$ is compact and dense in G. By compactness of $\widehat{F}_n$, the homomorphism $\hat{f}$ is open, so that G is a quotient of $\widehat{F}_n$.

If G is countably compact, then by a result of [D1], G is zero-dimensional. Therefore, the completion $\widehat{G} \in \mathcal{F}$ is a compact totally disconnected group. Since the above argument yields that $\widehat{G}$ is metrizable, we conclude that $G = \widehat{G}$.

(b) The quotient group $G/c(G) \in \mathcal{F}$ is countably compact and totally disconnected, so that the conclusion follows from (a). ☐

Let G be a topological group and $S \subseteq G \setminus \{1\}$. We call S a *supersequence converging to* 1 in G (denoted by $S \to 1$) if S is an infinite discrete subset of G and $S \cup \{1\}$ is compact (and hence homeomorphic to the one-point compactification of S).

2.2. Proposition. *Let G be a countably compact group. Then:*

(a) $G \in \mathcal{S}$ iff $G \in \mathcal{F}$ or there exists a supersequence $S \to 1$ in G such that $\overline{\langle S \rangle} = G$. Therefore, a countably compact group $G \in \mathcal{S}$ which is not topologically finitely generated (in particular, non-separable) must contain non-trivial convergent sequences.

(b) $G \in \mathcal{S}_c$ iff $G \in \mathcal{F}$. In such a case $G/c(G)$ is compact metrizable. Consequently, there are many compact (metrizable, totally disconnected) Abelian groups that do not belong to $\mathcal{S}_c$.

(c) $G \in \mathcal{S}_g$ iff G is finite.

Proof. (a) If the group H is topologically generated by a finite set, there is nothing to prove. Otherwise a suitable set S for G has to be infinite. One can assume that

$1 \notin S$. The complement $S \setminus U$ is closed discrete in G for each open neighborhood U of 1 in G, so that countable compactness of G implies that $S \setminus U$ is finite. This means that S is a supersequence converging to 1. The last assertion of (a) is obvious.

(b) The equivalence of $G \in \mathcal{S}_c$ and $G \in \mathcal{F}$ for a countably compact group G is immediate. The fact that $G/c(G)$ is compact metrizable for a countably compact group $G \in \mathcal{F}$ follows from Proposition 2.1 (b).

(c) If S is a generating suitable set for G, then $K = S \cup \{1\}$ is compact. Since $\langle K \rangle = G$, we conclude that G is σ-compact. Being countably compact, the group G has to be compact and, therefore, has the Baire property. Note that K is either finite or homeomorphic to the one-point compactification of the discrete space S, so that in any event K is scattered. Since $G = \langle K \rangle$, the group G is a countable union of continuous images of compact scattered spaces K^n, $n \in \mathbb{N}^+$. In other words, G is a countable union of compact scattered subspaces. Therefore, one of them must have a non-empty interior in G which in turn implies that G has an isolated point. Thus, G is compact discrete and hence is finite. □

2.3. Corollary. *Let $f: G \to H$ be a continuous surjective homomorphism of countably compact groups. If $G \in \mathcal{S}$ ($G \in \mathcal{S}_c$), then $H \in \mathcal{S}$ (resp., $H \in \mathcal{S}_c$).*

Proof. Suppose that $G \in \mathcal{S}$. If $G \in \mathcal{F}$ then obviously $H \in \mathcal{F}$. Otherwise, by Proposition 2.2 (a), there exists a supersequence S in G converging to 1_G such that $\langle S \rangle$ is dense in G. Therefore, both sets $S \cup \{1_G\}$ and $f(S) \cup \{1_H\}$ are compact, which implies that $S' = f(S) \setminus \{1_H\}$ is a supersequence in H converging to 1_H. It is clear that $\langle S' \rangle$ is dense in H, and hence $H \in \mathcal{S}$. □

Let us note that neither Proposition 2.2 (c) nor Corollary 2.3 can be extended to pseudocompact groups as Proposition 2.5 and Corollary 2.6 below show. We need the following simple lemma that appeared in [DTT]. For the sake of completeness we present its proof here.

2.4. Lemma. *Let G be a topological group.*
(a) If G contains a closed discrete subset A such that $|A| \geq d(G)$, then $G \times G$ has a closed suitable set or, equivalently, $G \times G \in \mathcal{S}_c$.
(b) If G contains a closed discrete subset A of size $|G|$, then $G \times G$ has a closed generating suitable set, that is, $G \times G \in \mathcal{S}_{cg}$.

Proof. (a) Let A be a closed discrete subset of G, $|A| \geq d(G)$. Choose a dense subset D of G with $|D| = d(G)$. Without loss of generality one can assume that the identity 1 of G is not in A. Denote by φ any map of A onto D. We define a subset S of $G \times G$ by

$$S = (A \times \{1\}) \cup (\{1\} \times A) \cup \{(x, \varphi(x)) : x \in A\} \cup \{(\varphi(x), x) : x \in A\}.$$

It is clear that S is closed discrete in $G \times G$ and the subgroup $\langle S \rangle$ of $G \times G$ generated by S contains $D \times \{1\}$ and $\{1\} \times D$, so that $\langle S \rangle$ is dense in $G \times G$.

(b) If A is a closed discrete subset of G and $|A| = |G|$, choose a bijection $\varphi: A \to G$ and define a subset S of $G \times G$ as in (a). This S will be a closed discrete generating set for $G \times G$. $\square$

2.5. Proposition. *There exist arbitrarily large pseudocompact Abelian groups in the class $\mathcal{S}_{cg}$.*

Proof. For each infinite cardinal τ, there exists a pseudocompact group G of cardinality 2^τ which contains a closed discrete subset of the same cardinality 2^τ. This follows from the fact that every completely regular space X can be embedded as a closed subspace into a pseudocompact group of cardinality not greater than $|X|^\omega \cdot \mathfrak{c}$. Indeed, a slight modification of the argument in [CSa, Theorem 2.4] shows that every precompact Abelian group embeds as a closed subgroup to a pseudocompact Abelian group satisfying the necessary cardinal restraint. In its turn, every Tikhonov space X is closed in the free Abelian topological group $A(X, \mathcal{V})$ relative to the variety $\mathcal{V}$ of precompact groups [Mo]. Thus, take a discrete space X of cardinality 2^τ and embed $A(X, \mathcal{V})$ as a closed subgroup to an appropriate pseudocompact Abelian group G of size 2^τ. Now, apply Lemma 2.4 (b) to conclude that the group $G \times G$ has a closed generating set, and hence $G \times G \in \mathcal{S}_{cg}$. By Comfort and Ross' theorem [CRs], the group $G \times G$ is pseudocompact. $\square$

The following result shows the difference between countably compact and pseudocompact groups with respect to taking quotients. Our construction will depend on the existence of a pseudocompact group $K \notin \mathcal{S}$ which is not countably compact (see Theorem 2.13 of [DTT]). We will also use the fact that a Σ-product $\Sigma(A) \subseteq \prod_{\alpha \in A} K_\alpha$ of pseudocompact topological groups K_α is pseudocompact. Indeed, the completion $\widehat{K}_\alpha$ of K_α is a compact group for each $\alpha \in A$ and one can easily verify that $\Sigma(A)$ intersects every non-empty G_δ-set in the compact group $\prod_{\alpha \in A} \widehat{K}_\alpha$, so that $\Sigma(A)$ is a pseudocompact group by Theorem 1.2 of [CRs].

2.6. Corollary. *There exist a pseudocompact Abelian group G and a closed pseudocompact subgroup N of G such that $G \in \mathcal{S}_{cg}$ but $G/N \notin \mathcal{S}$.*

Proof. Let $K \notin \mathcal{S}$ be a pseudocompact non countably compact topological group constructed in Theorem 2.12 of [DTT]. Then $nw(K) \leq \mathfrak{c}$, and hence $|K| \leq 2^{\mathfrak{c}}$. Since K is not countably compact, the group $H = K^{2^{\mathfrak{c}}}$ contains a closed copy of $\mathbb{N}^{2^{\mathfrak{c}}}$ and by a result of [Ju], H also contains a closed discrete subset P of cardinality $2^{\mathfrak{c}}$. Let $\Sigma = \Sigma(2^{\mathfrak{c}})$ be the Σ-product of $2^{\mathfrak{c}}$ many copies of the group K, $\Sigma \subseteq H$. It is clear that Σ is a dense pseudocompact subgroup of H and $|\Sigma| = 2^{\mathfrak{c}}$. Denote $L = \langle P \rangle + \Sigma$. Then L contains a dense pseudocompact subgroup Σ, so that L itself is a dense pseudocompact subgroup of H and $|L| = 2^{\mathfrak{c}}$. Since L contains a closed discrete subset P and $|P| = |L| = 2^{\mathfrak{c}}$, Lemma 2.4 (b) implies that $G = L \times L \in \mathcal{S}_{cg}$. Note that the group G is pseudocompact.

Let $p: L \times L \to L$ be the projection. Denote by φ the projection of $H = G^{2^{\mathfrak{c}}}$ to the first factor, $\varphi(x) = x_0$. It is clear that $\ker p \cong L$ is pseudocompact. The kernel Q of the homomorphism $\psi = \varphi|_L$ is also pseudocompact because Q contains a copy of the Σ-product $\Sigma(2^{\mathfrak{c}} \setminus \{0\})$ as a dense subgroup. Therefore, the kernel N of the homomorphism $q = \psi \circ p: L \times L \to K$ is topologically isomorphic to $Q \times L$ and hence is pseudocompact. Since both homomorphisms p and ψ are open, we conclude that $q: G \to K$ is also open, so that $K \cong G/N$. By the choice, $K \notin \mathcal{S}$, that completes the proof. □

In Section 3 the reader will find a detailed discussion of the stability problems for the classes $\mathcal{S}$, $\mathcal{S}_c$, etc.

Not every σ-compact topological group has a suitable set (see [C–T, Corollary 3.10] or Theorem 0.6). Let us establish certain conditions related with the existence of a closed (generating) suitable set for a σ-compact group.

2.7. Proposition. *Let G be a σ-compact group. Then:*

(a) If $G \in \mathcal{S}_c$ then G is separable; conversely, if G is separable and non-compact then $G \in \mathcal{S}_c$.

(b) $G \in \mathcal{S}_{cg}$ iff G is countable.

Proof. (a) Let $\bigcup_{n\in\omega} K_n = G \in \mathcal{S}_c$, with K_n being compact sets. If $S \subseteq G$ is a closed suitable set for G, then $S \cap K_n$ is closed discrete subset of the compact set K_n and hence is finite for each $n \in \omega$. This yields that S is countable. Suppose now that G is separable and non-compact. Being σ-compact, the group G is not pseudocompact. Therefore, Corollary 3.9 of [DTT] implies that $G \in \mathcal{S}_c$.

(b) Let $G \in \mathcal{S}_{cg}$ and suppose that $S \subseteq G$ is a closed generated suitable set for G. Then S is countable according to (a). Since $G = \langle S \rangle$, we conclude that G is countable as well. The converse follows from Theorem 2.2 of [C–T]. □

By Theorem 0.2 locally compact groups have suitable set. On the other hand, by Theorem 0.3 (b) the groups G with $d(G) < b(G)$ have closed suitable set. Now we see that this condition becomes also necessary for locally compact non-compact groups.

2.8. Proposition. *Let G be a non-compact locally compact group. Then $G \in \mathcal{S}_c$ iff $d(G) < b(G)$.*

Proof. As mentioned above we have to check only the necessity, so that let us assume $G \in \mathcal{S}_c$. If G is σ-compact then by the above proposition $d(G) \leq \omega$. On the other hand, being locally compact and non-compact, the group G cannot be precompact. Hence $b(G) > \omega$.

Assume now that G is not σ-compact. Let U be a compact neighbourhood of 1 in G. Then the subgroup H of G generated by U is open and σ-compact. As G is not σ-compact we have $[G : H] > \omega$. Let S be a closed suitable set of G. Fix $g \in G$. Repeating the argument of the first half of the proof of (a) in the above proposition we can conclude that the intersection $S \cap gH$ is countable. This yields $d(G) \leq \max\{|S|, \omega\} \leq [G : H]$. On the other hand, since H is open we have $[G : H] < b(G)$. This yields $d(G) < b(G)$. □

Local compactness is essential in Proposition 2.8. In fact, the subgroup $G = \mathbb{Q}/\mathbb{Z}$ of $\mathbb{T}$ is countable (hence σ-compact and separable) and precompact, thus $d(G) = b(G) = \omega$. By Theorem 0.4 $G \in \mathcal{S}_{cg}$.

The existence of non-trivial convergent sequences in infinite countably compact groups is an old and intriguing problem. Examples of infinite countably compact Abelian groups without non-trivial convergent sequences were given by Hajnal and Juhász [HJ] under CH and by van Douwen [vD1] under MA, but no ZFC example of such a group is constructed to the date. Let us see how the presence of a suitable set improves the situation.

2.9. Proposition. *An infinite countably compact torsion Abelian group $G \in \mathcal{S}$ contains non-trivial convergent sequences.*

Proof. Since every finitely generated subgroup of G is finite, a suitable set S for G has to be infinite; in particular $G \notin \mathcal{F}$. By Proposition 2.2 (a), there exists a supersequence $S \to 1$ in G, so that every countably infinite subset of S converges to 1. $\square$

One cannot omit "torsion" in Proposition 2.9. In fact, under CH there exists a countably compact monothetic Abelian group H without non-trivial convergent sequences. To see this, consider the free Abelian group G with $\mathfrak{c}$ generators. By a result of [Tk], the group G admits a countably compact, connected Hausdorff group topology $\mathcal{T}$ without non-trivial convergent sequences. Choose an element $x \in G \setminus \{0\}$ and denote by H the closure of $\langle x \rangle$ in $(G, \mathcal{T})$. It is clear that H is as required. The following example shows that the condition "$G \in \mathcal{S}$" in Proposition 2.9 is also essential.

2.12. Example. (See Theorem 3.15 of [C-T].) *There exists under MA an infinite separable countably compact subgroup G of $\{0,1\}^{\mathfrak{c}}$ without non-trivial convergent sequences. Therefore, G is a torsion group and $G \notin \mathcal{S}$.*

Example 2.12 shows that one cannot omit "countable pseudocharacter" condition in Theorem 0.3 (d). Theorem 2.4 (a) of [DTT] shows that "separable" in Theorem 0.3 (d) is also essential (there exists a Lindelöf non-separable topological group of countable pseudocharacter, by Proposition 2.13 below such a group cannot have a suitable set). Clearly a group as in Example 2.12 cannot be ω-bounded, since ω-bounded separable groups are compact. In Example 4.2.3 we give an example in ZFC of an ω-bounded minimal Abelian group without a suitable set.

As we have seen, compactness-like properties (such as countable compactness, σ-compactness, local compactness etc.) imply strict restrictions on the groups in $\mathcal{S}$ and $\mathcal{S}_c$. There is, however, at least one restraint for *all* topological groups in the class $\mathcal{S}$ as the following simple but useful result shows (see Lemma 2.3 of [DTT]).

2.13. Proposition. *A topological group $G \in \mathcal{S}$ satisfies $d(G) \leq L(G) \cdot \psi(G)$. In particular, a non-separable Lindelöf topological group of countable pseudocharacter does not have a suitable set.*

3. Invariance properties

3.1. Finer topologies, direct sums

We start with the following simple assertion:

3.1.1. Lemma. *$\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are closed under taking finer topologies.*

Proof. Discreteness and closedness are invariant under taking finer topologies. □

We do not know under which circumstances the conclusion of Lemma 3.1.1 remains true for the classes $\mathcal{S}$ and $\mathcal{S}_c$.

The following result is close to Theorem 4.1 of [C–T], but our construction of a suitable set is different because the method of [C–T] does not work for producing closed suitable sets for direct sums of topological groups. First, we need some notation.

Let $\Pi = \prod_{i \in I} G_i$ be a cartesian product of topological groups G_i. By definition, the *direct sum* $\oplus_{i \in I} G_i$ of the groups G_i is the σ-product of the groups G_i with the topology inherited from Π (see Section 1). Let A_i be a subset of G_i, $i \in I$. If $J \subseteq I$, we define

$$\oplus_{i \in J} A_i = \{x \in \oplus_{i \in I} G_i : x_i = 1_i \text{ for all } i \in I \setminus J \text{ and } x_i \in A_i \text{ for all } i \in J\}.$$

3.1.2. Theorem. *The classes $\mathcal{S}_c$ and $\mathcal{S}_{cg}$ are closed under taking countable direct sums.*

Proof. Let $\{G_i\}_{i \in \omega}$ be a family of topological groups, $G_i \in \mathcal{S}$ for each $i \in \omega$, and let $G = \oplus_{i \in \omega} G_i$. For every $i \in \omega$, denote by p_i the canonical projection $G \to G_i$, choose a closed discrete subset $S_i \subseteq G_i$ witnessing $G_i \in \mathcal{S}_c$ and define

$$A_i = \oplus_{j=0}^{i} S_i \quad \text{and} \quad S = \cup_{i \in \omega} A_i.$$

Then $S_i \times \{1_{\omega \setminus \{i\}}\} \subseteq \langle S \rangle$ for each $i \in \omega$, and hence $\overline{\langle S \rangle} = G$. Clearly, the set $B = \oplus_{i \in \omega}(S_i \cup \{1_i\})$ is closed in G, so that $\overline{S} \subseteq B$. Since the projection $p_0 \colon G \to G_0$ sends S onto the closed set S_0 in G_0 that misses 1_0, it follows that $1_0 = p_0(1) \notin \overline{p_0(S)}$. Consequently, $1 \notin \overline{S}$ in G. Let us show that S is closed discrete in G.

Take a point $1 \neq g = (g_i)_{i \in \omega} \in \overline{\langle S \rangle}$. Then $g \in \oplus_{j \leq i} G_j$ for some $i \in \omega$. Since $g \neq 1$ we can choose i to be minimal, that is, $g_i \neq 1_i$. These assumptions give $g \in B \cap \oplus_{j \leq i} G_j = \oplus_{j \leq i}(S_j \cup \{1_j\})$. Since $g_{i+1} = 1_{i+1} \notin \overline{S_{i+1}}$, there exists an open neighborhood U_{i+1} of 1_{i+1} in G_{i+1} such that $U_{i+1} \cap S_{i+1} = \emptyset$. This immediately yields that $W \cap A_m = \emptyset$ for every $m > i$, where $W = p_{i+1}^{-1}(U_{i+1})$ is an open neighborhood of g in G. This implies that $g \in \overline{A_1 \cup \ldots \cup A_i}$. Since $p_i(\overline{A_k}) = \{1_i\}$ for some $k < i$, we conclude that $g \notin \overline{A_k}$ for $k < i$. Hence $g \in \overline{A_i}$. The latter is equivalent to say that $(g_0, \ldots, g_i) \in \overline{S_1 \times \ldots \times S_i}$. However, the subset $S_1 \times \ldots \times S_i$ of $G_0 \times \ldots \times G_i$ is closed and discrete as a finite product of closed discrete sets. Thus, $g \in A_i \subseteq S$ and g is an isolated point of S. Hence S is a closed discrete subset of G and $G \in \mathcal{S}_c$.

If each S_i generates G_i then S generates $\oplus_{i \in \omega} G_i$. Therefore, $G \in \mathcal{S}_{cg}$. This proves the lemma. $\square$

It will be proved in Corollary 3.2.4 that $\mathcal{S}$ and $\mathcal{S}_g$ are closed under taking arbitrary direct sums. However, this is not true for $\mathcal{S}_c$ and $\mathcal{S}_{cg}$ even in the case of direct sums of ω_1 finite cyclic groups as the following example shows.

3.1.3. Example. *The direct sum $G = \oplus_{\omega_1}\{0,1\}$ of ω_1 copies of the discrete group $\{0,1\}$ does not belong to $\mathcal{S}_c$. Indeed, the group G is σ-compact as the subset K_n of all elements of G with support of size $\leq n$ is compact for every $n \in \mathbb{N}$. Since $d(G) > \aleph_0$, Proposition 2.7(a) implies that $G \notin \mathcal{S}_c$.*

Let us finish this section with one more example. The group $\mathbf{J}_p$ of p-adic integers is monothetic, and hence the direct sum $G = \oplus_\omega \mathbf{J}_p$ of countably many copies of $\mathbf{J}_p$ belongs to $\mathcal{S}_c$ by Theorem 3.1.2. Nevertheless, $G \notin \mathcal{S}_{cg}$, for G is σ-compact and uncountable (see Proposition 2.7 (b)).

3.2. Products

The following result is immediate.

3.2.1. Proposition. *Let K be a dense subgroup of a topological group G and let S be a suitable set for G with $S \subseteq K$. Then S is a suitable set for every group H satisfying $K \subseteq H \subseteq G$.*

The part (a) of the following lemma slightly generalizes Theorem 4.1 of [C–T]; the part (b) is new.

3.2.2. Lemma. *Let H be a subgroup of a cartesian product $G = \prod_{i \in I} G_i$ of topological groups G_i. Suppose that for each $i \in I$, $p_i(H)$ contains a subgroup L_i such that $L = \oplus_{i \in I} L_i$ is dense in H, where $p_i \colon G \to G_i$ is the projection. Then:*

(a) if for every $i \in I$, $p_i(H)$ contains a suitable set S_i with $S_i \subseteq L_i$, then $H \in \mathcal{S}$;

(b) $L \in \mathcal{S}_g$ whenever $L_i \in \mathcal{S}_g$ for each $i \in I$.

Proof. (a) Let S_i be a suitable set for $K_i = p_i(H)$ satisfying $S_i \subseteq L_i$, $i \in I$. Define $S = \bigcup_{i \in I}(S_i \times \{1_{I \setminus \{i\}}\}) \subseteq L$ and use the argument in [C–T, Theorem 4.1] to show that S is suitable for $\prod_{i \in I} K_i$ and, by Proposition 3.2.1, for H and L.

(b) For every $i \in I$, choose a suitable set S_i for L_i with $\langle S_i \rangle = L_i$. By (a), the set $S = \bigcup_{i \in I}(S_i \times \{1_{I \setminus \{i\}}\})$ is suitable for L. The equality $\langle S \rangle = L$ is immediate. $\square$

The first part of the following result was proved in [C–T, Theorem 4.1].

3.2.3. Corollary. *The class $\mathcal{S}$ is closed under arbitrary products. More precisely, if $G_i \in \mathcal{S}$ for every $i \in I$, then every subgroup H of the cartesian product $G = \prod_{i\in I} G_i$ containing the direct sum $\oplus_{i\in I} G_i$ belongs to $\mathcal{S}$.*

3.2.4. Corollary. *The classes $\mathcal{S}$ and $\mathcal{S}_g$ are closed under taking arbitrary direct sums.*

Corollary 3.2.3 implies that $\mathcal{S}$ is closed under taking Σ-products (this was also noted in [C–T, Theorem 4.1]). We see now that this is not true either for $\mathcal{S}_c$ or $\mathcal{S}_{cg}$. In fact, if the index set I is infinite, the set S in the above proof has the identity 1 as a cluster point, so that S chosen in that way cannot be closed. It turns out that no closed suitable set can exist in some cases.

3.2.5. Proposition. *The classes $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are closed under finite products but fail to be closed under infinite products.*

Proof. The case of finite products follows from Theorem 3.1.2 and Lemma 3.2.2. For infinite products we shall give a single counterexample which serves for all three classes. The key is the following corollary to Proposition 2.2: if a compact group G is in some of these three classes then $G \in \mathcal{F}$. Let C be a finite cyclic group, $|C| > 1$ (so that $C \in \mathcal{S}_{cg}$), and put $G = C^{\omega}$. Then every finitely generated subgroup of G is finite, whence $G \notin \mathcal{F}$. Therefore, items (b) and (c) of Proposition 2.2 imply that $G \notin \mathcal{S}_c$ and $G \notin \mathcal{S}_g$. □

3.3. Subgroups

The following example from [C–T, Theorem 4.7] shows that *open* subgroups of groups in $\mathcal{S}$ need not belong to $\mathcal{S}$.

3.3.1. Example. (See [C–T].) *For every topological group G, the group $H = G \times G_d$ belongs to $\mathcal{S}_{cg}$, where G_d denotes the group G equipped with the discrete topology. So, if $G \notin \mathcal{S}$ then the projection $p: H \to G$ is an open homomorphism which does not preserve the classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$.*

This example shows that the property being in $\mathcal{S}$ (or in $\mathcal{S}_c$, $\mathcal{S}_g$, $\mathcal{S}_{cg}$) is inherited neither by open subgroups, nor by direct summands. The proof of the fact that $H = G \times G_d$ belongs to $\mathcal{S}_{cg}$ follows immediately from the following claim. The first part can be easily deduced from Theorem 0.3 (b).

Claim. *If G has an open subgroup H such that $d(H) \leq [G : H]$, then $G \in \mathcal{S}_c$. If $|H| \leq [G : H]$, then $G \in \mathcal{S}_{cg}$.*

Now we discuss the passage to *dense* subgroups. Let us recall that dense subgroups of metrizable topologically finitely generated groups are in the class $\mathcal{S}_c$ (see the argument after (c) of Introduction for a direct proof; this also follows from Theorem 0.3 (a)). It was noted in [C–T, Remark 3.13] that $\mathcal{S}$ is not closed under taking dense subgroups: the monothetic group $\mathbb{T}^{\mathfrak{c}}$ contains a dense σ-compact connected subgroup $H \notin \mathcal{S}$. On the other hand, a group G with a dense $\mathcal{S}$-subgroup not necessarily has a suitable set (see Example 2.13 of [DTT]). Therefore, the class of groups containing a dense $\mathcal{S}$-subgroup is strictly larger than $\mathcal{S}$.

It is a challenging problem to characterize the latter class. We just note here that under $\diamondsuit$ this class of groups does not coincide with the class of all topological groups (see Theorem 2.4 (b) of [DTT]). We do not know, however, if there exists in ZFC an example of a topological group which does not contain a dense subgroup with a suitable set (see Problem 3.18 of [DTT]).

The counterpart for closed $\mathcal{S}$-subgroups may be put as follows: Does a given group G have a reasonably large closed $\mathcal{S}$-subgroups? The answer is "no" under MA: the only closed $\mathcal{S}$-subgroups of the group G in Example 2.12 are the finite ones. This suggests to restrict the class of closed subgroups. In particular, it seems interesting to find out whether homomorphic retracts of groups in $\mathcal{S}$ ($\mathcal{S}_c$) belong to $\mathcal{S}$ ($\mathcal{S}_c$).

3.4. Quotients

We begin with the question: *when is $\mathcal{S}$ closed under taking quotients?* The theorem below generalizes the fact that continuous surjective homomorphisms with compact domain preserve $\mathcal{S}$ "downwards" (i.e., if G is compact then every quotient of G is in $\mathcal{S}$). We have seen in Corollary 2.3 that such a preservation is available also when the domain is countably compact.

3.4.1. Theorem. *The classes $\mathcal{S}$, $\mathcal{S}_c$, $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are closed under taking closed homomorphic images.*

Proof. Let $f: G \to H$ be a closed continuous surjective homomorphism. Suppose that S is a suitable set for G. Then $\overline{S} \subseteq S \cup \{1_G\}$. Define $S_1 = f(S) \setminus \{1_H\}$. Since f is closed, we have $S_1 \cup \{1_H\} = f(S \cup \{1_G\})$ and the latter set is closed in H. An analogous argument applied to an arbitrary subset S' of S shows that S_1 is discrete and the identity 1_H can be the only accumulation point of S_1, i.e., S_1 is suitable for H. It remains to note that if S is closed (generating), then S_1 has the same property. □

3.4.2. Corollary. *If $G \in \mathcal{S}$ and K is a compact normal subgroup of G then $G/K \in \mathcal{S}$. The same is true for the classes $\mathcal{S}_c, \mathcal{S}_g$ and $\mathcal{S}_{cg}$.*

Example 3.3.1 shows that $G \in \mathcal{S}_{cg}$ and $K \in \mathcal{S}_{cg}$ do not imply $G/K \in \mathcal{S}$ even when K is discrete, i.e., closedness of f is essential in Theorem 3.4.1.

In what follows we investigate the preservation of $\mathcal{S}$ in the opposite direction. Namely, we give some partial results about "lifting" the property being in $\mathcal{S}$ (or in $\mathcal{S}_c$, $\mathcal{S}_g$, $\mathcal{S}_{cg}$) along a "nice" surjective homomorphism $f: G \to H$, i.e., we investigate when $H \in \mathcal{S}$ yields $G \in \mathcal{S}$ and similarly for the other classes. According to Lemma 3.1.1 it is not restrictive to consider only open homomorphisms f when $\mathcal{S}_g$ and $\mathcal{S}_{cg}$ are involved. It is also natural to assume that $\ker f \in \mathcal{S}$. So, our general problem can be reformulated as follows:

3.4.3. Problem. *Let N be a closed normal subgroup of a topological group G such that $G/N \in \mathcal{S}$ and $N \in \mathcal{S}$.*

(a) Is it true then that $G \in \mathcal{S}$?

(b) What is the answer to (a) if N is either compact or metrizable, or discrete ?

By Proposition 3.2.5 and Corollary 3.2.3, the answer to Problem 3.4.3 (a) is "yes" when N is a topological direct summand, i.e., $G \cong N \times G/N$. We show that this remains true for semidirect products as well (see Corollary 3.4.8). In the sequel we shall develop a line "transversal" to (b), i.e., we shall impose some additional restrictions on the *quotient* G/N in order to get $G \in \mathcal{S}$. For example, Theorem 4.2 of [C–T] answers Problem 3.4.3 (a) affirmatively for both $\mathcal{S}$ and $\mathcal{S}_c$ when N is open, i.e., when G/N is discrete. Our first aim will be to extend this observation to the more general case when $G/N \in \mathcal{S}_c$ (see Theorem 3.4.5).

The proof of the following simple lemma is omitted.

3.4.4. Lemma. *Suppose that $f\colon G \to H$ is a continuous homomorphism, $S \subseteq G$, $f(S) = S_1$ and the restriction $f|_S\colon S \to S_1$ is one-to-one. Then:*

(a) if S_1 is discrete then S is discrete;

(b) if S_1 is closed and discrete in H then S is closed and discrete in G.

3.4.5. Theorem. *Let N be a closed normal subgroup of topological group G such that $H = G/N \in \mathcal{S}_c$. Then:*

(a) if $N \in \mathcal{S}$ then $G \in \mathcal{S}$;

(b) if $N \in \mathcal{S}_c$ then $G \in \mathcal{S}_c$.

Proof. Let $S_1 \subseteq H \setminus \{1_H\}$ be a closed discrete subset of H such that $\langle S_1 \rangle$ is dense in H. Choose a subset S of G such that $f(S) = S_1$ and $f|_S$ is one-to-one. Then S is closed discrete in G by Lemma 3.4.4 and it is clear that $S \cap N = \emptyset$.

(a) Assume that $N \in \mathcal{S}$ and let S_2 be a suitable set for N. Then $S_3 = S \cup S_2$ satisfies $\overline{S_3} = \overline{S} \cup \overline{S_2} = S \cup \overline{S_2}^N$ because S and N are closed in G. In addition, by the choice of S_2 we have $\overline{S_2}^N \subseteq S_2 \cup \{1_G\}$. This proves that $\overline{S_3} \subseteq S_3 \cup \{1_G\}$. Since both S and S_2 are discrete and the set $\overline{S_2}^N \subseteq N$ is disjoint from S, we conclude that S_3 is discrete. Finally, the subgroup $L = \overline{\langle S_3 \rangle}$ of G contains N, and hence $f(L)$ is closed in H where $f\colon G \to H$ is the quotient homomorphism. Since $f(L)$ contains the dense subgroup $\langle S_1 \rangle$ of H, we conclude that $f(L) = H$. Therefore, $L = f^{-1}f(L) = G$.

(b) Assume that $N \in \mathcal{S}_c$ and let $S_2 \subseteq H$ be a closed discrete subset witnessing $H \in \mathcal{S}_c$. Argue as above and note that the set S_3 is closed in this case. $\square$

3.4.6. Remark. Using the argument of Theorem 3.4.5, one can prove that $G/N \in \mathcal{S}_{cg}$ in conjunction with $H \in \mathcal{S}_g$ (resp., $H \in \mathcal{S}_{cg}$) yields $G \in \mathcal{S}_g$ (resp., $G \in \mathcal{S}_{cg}$).

Now we present another condition to ensure the positive answer to Problem 3.4.3 (a). We shall say that a homomorphism $f : G \to H$ *has a closed section* if there exists a closed subset $X \subseteq G$ such that the restriction of f to X is injective and $f(X) = H$, i.e., there exists a (not necessarily continuous) section $s: H \to G$ such that $X = s(H)$ is closed in G.

3.4.7. Lemma. *Let $f: G \to G/N$ be a quotient homomorphism with a closed section and $N \in \mathcal{S}$. Then $G/N \in \mathcal{S}$ implies $G \in \mathcal{S}$.*

Proof. Assume that S_1 is a suitable set for $H = G/N$ such that $1_H \notin S_1$. Fix a closed section $\sigma: H \to G$ for f. Without loss of generality one can assume that $1_G \in \sigma(H)$ (otherwise choose $a \in \sigma(H) \cap N$ and define a closed section $\sigma^*: H \to G$ by $\sigma^*(y) = a^{-1} \cdot \sigma(y)$ for every $y \in H$). Choose a subset $S \subseteq \sigma(H)$ such that $f(S) = S_1$. The set S is discrete by Lemma 3.4.4. It is clear that $\overline{S} \subseteq \sigma(H) \cap f^{-1}(\overline{S_1})$. Since $\overline{S_1} \subseteq S_1 \cup \{1_H\}$ and $f|_{\sigma(H)}$ is one-to-one, we have $\overline{S} \subseteq S \cup \{1_G\}$. Fix a suitable set S' for N with $1_G \notin S'$. Then $S_2 = S \cup S'$ is suitable for G. Indeed, $\overline{S_2} \subseteq S_2 \cup \{1_G\}$ and S_2 is discrete because $1_G \notin S_2$ and the sets S and S' are discrete, disjoint and $\overline{S} \subseteq S \cup \{1_G\}$, $\overline{S'} \subseteq S' \cup \{1_G\}$. Finally, the group $L = \overline{\langle S_2 \rangle}$ contains N, so that $f(L)$ is a closed subgroup of G/N. As in the proof of Theorem 3.4.5, this yields $f(L) = G/N$, and hence $L = G$. $\square$

3.4.8. Corollary. *$\mathcal{S}$ is closed under the semidirect product operation.*

3.4.9. Corollary. *If G is a connected pseudocompact group and $N \in \mathcal{S}$ (resp. $N \in \mathcal{S}_c$) is a closed normal G_δ-subgroup of G, then $G \in \mathcal{S}$ (resp. $G \in \mathcal{S}_c$).*

Proof. By Theorem 3.2 of [CRn], G/N is a compact connected metrizable group, whence $G/N \in \mathcal{F} \subseteq \mathcal{S}_c$. Now Theorem 3.4.5 applies. $\square$

Note that a compact group belonging to $\mathcal{S}_g$ is finite by Proposition 2.2 (c), so that one can trivially extend the above corollary to the classes $\mathcal{S}_g$ and $\mathcal{S}_{cg}$.

4. Minimal groups with suitable sets

4.1. Totally minimal groups

Here we show that the totally minimal groups almost always have suitable sets (see Problem 4.1.6 below). Let us note that total minimality is the only example, apart from local compactness, of a compact-like property that may entail the existence of suitable sets: none of the compact-like properties considered so far (ω-boundedness, completeness, being Lindelöff, σ-compactness) implies the existence of a suitable set.

4.1.1. Lemma. *Let $f: G \to G_1$ be a continuous surjective homomorphism of compact groups, H_1 a dense subgroup of G_1. Then the subgroup $H = f^{-1}(H_1)$ of G is dense and:*
(a) $H_1 \cong H/\ker f$;
(b) H is totally minimal if H_1 is totally minimal;
(c) H is minimal if H_1 is minimal;
(d) H is countably compact if H_1 is countably compact;
(e) H is ω-bounded if H_1 is ω-bounded;
(f) $H_1 \in \mathcal{S}$ (resp., $H_1 \in \mathcal{S}_c$, $\mathcal{S}_g$, $\mathcal{S}_{cg}$) if $H \in \mathcal{S}$ (resp., $H \in \mathcal{S}_c$, $\mathcal{S}_g$, $\mathcal{S}_{cg}$).

Proof. The density of H and (a) are obvious, while (b)–(d) are well known (for instance, (b) and (e) are proved in [DS, Lemma 2.1], (c) can be found in [D2] and (d) is proved in [CRn]). For (f), one can apply Corollary 3.4.2 along with the fact that $\ker f$ is compact. $\square$

4.1.2. Lemma. *If H is a dense totally minimal subgroup of a product $K = \prod_{i \in I} K_i$ of metrizable groups K_i, then $H \in \mathcal{S}$.*

Proof. For each $i \in I$, identify the factor K_i with the subgroup $K_i \times \{1_{I \setminus \{i\}}\}$ of K. By the total minimality criterion [DP] (see also [DPS, Theorem 4.3.3]), for every $i \in I$ the subgroup $L_i = K_i \cap H$ is dense in K_i. Since each K_i is metrizable, Corollary 3.8 of [DTT] implies that for every $i \in I$, K_i contains a suitable set S_i with $S_i \subseteq L_i$. It is clear that $\oplus_{i \in I} L_i$ is dense in $\prod_{i \in I} L_i$, so that $\oplus_{i \in I} L_i$ is dense in H. Thus, $H \in \mathcal{S}$ by Lemma 3.2.2 (a). $\square$

Our study of totally minimal groups depends on the following fact which is important in itself. In the Abelian case, the same result was earlier established by Hoffman and Morris [HM, Lemma 1.6].

4.1.3. Theorem. *Let G be a compact topological group which is either Abelian or connected. Then G is a quotient group of a cartesian product $\prod_{i \in I} K_i$ of compact metrizable groups. In addition, if G is Abelian (connected), then the factors K_i can be chosen Abelian (connected). If G is Abelian and totally disconnected, then the factors K_i can also be chosen Abelian and totally disconnected.*

Proof. (1) First, suppose that G is Abelian. Take the discrete Pontryagin dual group X for G and its divisible hull $D(X)$ (see Theorem 4.1.6 of [Ro]). Then $D(X) = \oplus_{i \in I} D_i$ where each D_i is a countable divisible group isomorphic to $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ for a prime p. Dualizing the inclusion homomorphism $X \hookrightarrow D(X)$, we get the desired continuous surjective homomorphism $f : \prod_{i \in I} K_i \to G$, where K_i is the compact Pontryagin dual group for D_i. Note that if G is connected then X and $D(X)$ are torsion-free [HR, Theorem 24.25], so that each D_i is isomorphic to $\mathbb{Q}$. Therefore, the dual K_i for D_i is connected for all $i \in I$.

If G is totally disconnected then X is a torsion group [HR, Theorem 24.26], so that $D(X)$ is also a torsion group. This implies that each summand D_i is isomorphic to $\mathbb{Z}(p^\infty)$ for a prime p, and hence the compact dual group K_i for D_i is zero-dimensional [HR, Theorem 24.21].

(2) Now let G be an arbitrary compact connected group. By a theorem of Varopoulos [V], G is a quotient of $A \times L$, where A is the connected component of the center $Z(G)$ of G and L is a product of compact connected Lie groups. By (1), we can represent the compact connected Abelian group A as a quotient of a product $\prod_{j\in J} K_j$ of compact, connected, metrizable Abelian groups K_j. Now $L \times \prod_{j\in J} K_j$ works. $\square$

Note that Theorem 4.1.3 immediately implies that compact Abelian groups and compact connected groups are dyadic.

4.1.4. Theorem. *Every totally minimal Abelian group has a suitable set.*

Proof. Let G be a totally minimal Abelian group. Then the completion $\widehat{G}$ of G is a compact Abelian group by Prodanov's precompactness theorem [P]. By Theorem 4.1.3, there exists a continuous surjective homomorphism $f\colon K \to \widehat{G}$, where $K = \prod_{i\in I} K_i$ and every K_i is a compact metrizable group. According to Lemma 4.1.1, $H = f^{-1}(G)$ is a dense totally minimal subgroup of K and $H/N \cong G$, where N is the kernel of f. From Lemma 4.1.2 it follows that $H \in \mathcal{S}$. Now Lemma 4.1.1 (f) completes the proof. $\square$

The following theorem shows that the class $\mathcal{S}$ contains all totally minimal connected precompact groups.

4.1.5. Theorem. *Every totally minimal connected precompact group G has a suitable set.*

Proof. Since the completion $\widehat{G}$ of G is a compact connected group, we can find a continuous surjective homomorphism $f\colon \prod_{i\in I} K_i \to \widehat{G}$, where every K_i is a compact metrizable group. It remains to apply the scheme of the above proof based essentially on Lemma 4.1.1 to obtain $G \in \mathcal{S}$. $\square$

4.1.6. Problem. *Does $\mathcal{S}$ contain all totally minimal groups? What about precompact totally minimal groups ?*

It is worth mentioning that most of the known totally minimal non-precompact (hence, non-Abelian) groups are either locally compact or products of locally compact groups (see [D2, §3.1], or [DPS, Chap. 7]), and hence belong to $\mathcal{S}$ by our Corollary 3.2.3 and Theorem 1.12 of [HM] (see Theorem 0.2 (a)).

We finish with another example of a non-precompact totally minimal group supporting the positive answer to Problem 4.1.6.

4.1.7. Example. *Let X be an infinite set and $G = S(X)$ the symmetric group equipped with the topology of pointwise convergence. Then $G \in \mathcal{S}_c$.*

Proof. Choose a point $x_0 \in X$ and define $N = \{f \in G : f(x_0) = x_0\}$. Then N is an open subgroup of G, G/N is discrete and $|G/N| = |X|$. Therefore, $b(G) = |X|^+ > d(G) = |X|$, so that Theorem 5.7 of [C–T] (see Theorem 0.3 (b)) applies. $\square$

The totally minimality of the group $S(X)$ was established by Dierolf and Schwanengel [DiSw].

4.2. Minimal groups

Example 4.2.3 below shows that a minimal countably compact Abelian group need not have a suitable set. However, connectedness helps as the following result shows.

4.2.1. Theorem. *Every minimal countably compact connected Abelian group G has a suitable set.*

Proof. The completion $\widehat{G}$ of G is a quotient of a power of the compact connected metrizable group $K = Hom(\mathbb{Q}, \mathbb{T})$, say K^I (see Theorem 4.1.3). According to the scheme described in the proof of Theorem 4.1.4, it suffices to assume that G is a dense minimal countably compact (necessarily connected) subgroup of K^I. Then one can show (as in [D3]) that there exists a finite subset A of I such that the group G contains the Σ-product of the groups $K_i = K$, with $i \in I \setminus A$. Define $K_1 = K^{I \setminus A} \times \{0_A\}$ and $G_1 = G \cap K_1$. Since $K \in \mathcal{S}$, Corollary 3.2.3 implies that $G_1 \in \mathcal{S}$. On the other hand, G_1 is a closed G_δ-subgroup of the connected countably compact group G, so that we can apply Corollary 3.4.9 to conclude that $G \in \mathcal{S}$. $\square$

We conjecture that "Abelian" can be eliminated from Theorem 4.2.1. On the other hand, the topological assumptions can be weakened as follows: *every connected Abelian group that contains a dense minimal countably compact group has a suitable set.* Note that every group that contains a dense minimal subgroup is minimal, but the counterpart for countable compactness does not hold true in general. Hence this provides a larger class of groups (the analogous property in the category of topological spaces is known as *strong pseudocompactness*).

4.2.2. Theorem. *Let G be a countably compact minimal Abelian group. Then both $c(G)$ and $G/c(G)$ are minimal and $c(G) \in \mathcal{S}$. If $G \in \mathcal{S}$, then also $G/c(G) \in \mathcal{S}$.*

Proof. As a closed subgroup of G the subgroup $c(G)$ is minimal ([DPS, Proposition 2.5.7]). Now Theorem 4.2.1 implies $c(G) \in \mathcal{S}$. Since the group G is countably compact, $c(G)$ coincides with the quasicomponent of G and the quotient $G/c(G)$ is zero-dimensional ([D1, Theorem 1.2]). Now the minimality of G implies that also $G/c(G)$ is minimal ([D1, Theorem 1.7]). The last assertion follows from Corollary 2.3. $\square$

Let us note that the above theorem settles one of the directions in the reduction of the study of the countably compact minimal Abelian groups in $\mathcal{S}$ to case of totally disconnected ones. In the opposite direction one has to be able to conclude that if

$G/c(G) \in \mathcal{S}$ (and $c(G) \in \mathcal{S}$, granted by Theorem 4.2.1) then $G \in \mathcal{S}$. To this end one needs the positive answer of Problem 3.4.3 (b) (at least in the case of a compact subgroups, since $c(G)$ is compact for a countably compact minimal Abelian group G whenever $|c(G)|$ is measurable [D3]).

4.2.3. Example. *There exists a totally disconnected ω-bounded (and hence countably compact) minimal Abelian group without suitable sets.*

Proof. Let G be the ω-bounded dense subgroup of $\mathbb{Z}(2)^{\mathfrak{c}}$ with $G \notin \mathcal{S}$ constructed in Theorem 2.12 of [DTT]. The group G is not minimal (since a minimal group of exponent 2 must be compact, cf. [DPS, Example 2.5.3 (b)]). In order to get a minimal group we consider a surjective continuous homomorphism $f : K \to \mathbb{Z}(2)^{\mathfrak{c}}$ where K is a totally disconnected compact Abelian group. Then by Lemma 4.1.1 (e), (f) the subgroup $H = f^{-1}(G)$ of K is ω-bounded and $H \notin \mathcal{S}$. In addition, H is totally disconnected as a subgroup of K. To ensure minimality of H one has to take f in such a way that every closed non-trivial subgroup of K meets $\ker f$ (cf. [DPS, Theorem 2.5.1]). There are many choices of such an f: (i) $K = \mathbb{Z}(4)^{\mathfrak{c}}$ and f is the homomorphism "multiplication by 2", (ii) $K = \mathbf{J}_2^{\mathfrak{c}}$ and $\ker f = 2K$. In the case (i) H has exponent 4, in the case (ii) H is torsion-free. ☐

5. Suitable sets in Bohr topology

The *Bohr topology* of a topological group G is the topology of G induced by the Bohr compactification $r_G : G \to bG$. In general, the homomorphism r_G need not be injective. For a locally compact Abelian group G the homomorphism r_G is injective and we denote by G^+ the group G equipped with the Bohr topology. In the special case when G is an Abelian group with the discrete topology, we follow van Douwen [vD2] and write $G^\#$ in place of G^+. An equivalent description of G^+ is as follows: G^+ is the group G equipped with the initial topology with respect to all *continuous* homomorphisms of G to $\mathbb{T}$. Every homomorphism $f : G^\# \to H^\#$ is continuous. More generally, if G, H are locally compact Abelian groups and $f : G \to H$ is a continuous homomorphism, then also $f : G^+ \to H^+$ is continuous. In other words, $^+$ is a functor from the category of locally compact Abelian groups to the category of precompact abelian groups. Actually, $^+$ is a reflector defined on the larger class of *maximally almost periodic groups*, namely *all* groups G such that the homomorphism r_G is injective, with values in the category all precompact groups. For the restricted purposes of our exposition we prefer to apply this functor only to locally compact Abelian groups and we give in the theorem below some useful properties of the group $G^\#$ and the functors $^+$ and $^\#$ (the only case when local compactness may fail will be the part of item (d) regarding infinite products).

5.1. Theorem. *(a)* $\chi(G^\#) = w(G^\#) = 2^{|G|}$, $d(G^\#) = |G|$, $\psi(G^\#) = \log |G|$ *and* $b(G^\#) = \omega$ *for every infinite Abelian group* G.

(b) The groups G *and* G^+ *have the same compact sets. Hence the only compact*

subsets of $G^\#$ are the finite ones. In particular, $G^\#$ has no convergent sequences beyond the trivial ones.

(c) The functor $^+$ "respects" (in the spirit of (b)) other compactness-like properties: pseudocompactness, realcompactness, Lindelöff property, etc. Therefore, for an infinite group G, $G^\#$ is never pseudocompact.

(d) The functor $^+$ preserves topological subgroups, quotients and direct products.

(e) All subgroups of $G^\#$ are closed.

(f) If S is an independent subset of G then S is closed and discrete in $G^\#$.

(g) $G^\#$ contains a closed discrete subset of size $|G|$.

Proof. (a) The first two equalities follow from the well known equalities $\chi(bG) = w(bG) = 2^{|G|}$. The equality $d(G^\#) = |G|$ follows from (h) (whose proof does not depend on (a)–(g)). The equality $b(G^\#) = \omega$ simply says that $G^\#$ is precompact.

To prove the equality $\psi(G^\#) = \log |G|$ suppose that G is uncountable (otherwise the equality is trivially satisfied). Note that $\psi(G^\#)$ coincides with the minimum number $|I|$ of homomorphisms $f_i : G \to \mathbb{T}$, $i \in I$, that separate the points of G. For every point separating family $\{f_i\}_{i\in I}$, the diagonal product $\Delta_{i\in I} f_i$ is an injective homomorphism of G to $\mathbb{T}^I$, which gives the inequality $|G| \leq |\mathbb{T}^I| = 2^{|I|}$ and, consequently, $|I| \geq \log |G|$. On the other hand, since $\mathbb{T}$ is divisible and contains a copy of every finite cyclic group, every Abelian group G is a subgroup of some power $\mathbb{T}^I$, where I has to be taken so that the obvious inequalities $r(G) \leq r(\mathbb{T}^I) = 2^{|I|}$ and $r_p(G) \leq r_p(\mathbb{T}^I) = 2^{|I|}$ be satisfied (here r is the free-rank and r_p is the p-rank). Since $|G| = \max\{r(G), \sup_p\{r_p(G)\}\}$ gives $|G| \leq 2^{|I|}$, one can choose I with the only restraint $|I| \geq \log |G|$. This proves $\psi(G^\#) = \log |G|$.

(b) See Glicksberg's article [Gl].

(c) All these facts are proved by Trigos-Arrieta [Tr].

(d) Let H be a closed subgroup of a locally compact Abelian group G. We have to prove that the subgroup H^+ is a topological subgroup of G^+. The continuity of the inclusion $\iota : H^+ \to G^+$ follows from the functoriality of $^+$. To see that ι is an embedding recall that every continuous character $H \to \mathbb{T}$ extends to a continuous character $G \to \mathbb{T}$ ([HR, 24.12]).

Now we show that $(G/H)^+ = G^+/H^+$, where the latter group is equipped with the quotient topology. The continuity of the quotient homomorphism $f : G^+ \to (G/H)^+$ and the identity $(G/H)^+ \to G^+/H^+$ follows from the functoriality of $^+$. The continuity of the identity $G^+/H^+ \to (G/H)^+$ follows from the continuity of the quotient homomorphism f and the universal property of the quotient topology. The proves the preservation of quotients under $^+$.

Let G, H be locally compact Abelian groups. The continuity of the identity map $(G \times H)^+ \to G^+ \times H^+$ follows easily from the functoriality of $^+$. Again by functoriality of $^+$, both inclusions $i: G^+ \hookrightarrow (G \times H)^+$ and $j: H^+ \hookrightarrow (G \times H)^+$ are continuous. Since the product topology of $G^+ \times H^+$ has the universal property with respect to the monomorphisms i and j ([DPS, Exercise 2.10.4 (i)]), it follows that the identity $G^+ \times H^+ \to (G \times H)^+$ is continuous. This proves the preservation of finite direct products under $^+$. The case of arbitrary direct products may lead out of the category of locally compact groups. Nevertheless, the preservation is still available since Bohr compactification commutes with arbitrary direct products ([HdV]).

(e) Let H be a subgroup of G. Note that the homomorphisms of G/H to $\mathbb{T}$ separate the points of G/H. In view of the properties from (d) this yields that H is a closed subgroup of $G^\#$.

(f) and (g) are proved in Lemma 1.4 and Proposition 2.5 of [HvM2]. ☐

We will also use the following simple fact.

5.2. Proposition. *Let G be an Abelian group. Then:*

(i_1) $G^\# \in \mathcal{S}$ iff $G^\# \in \mathcal{S}_g$; (i_2) $G^\# \in \mathcal{S}_c$ iff $G^\# \in \mathcal{S}_{cg}$.

For an arbitrary precompact group topology τ for G, we have:

(ii_1) if $(G,\tau) \in \mathcal{S}_g$ then $G^\# \in \mathcal{S}_g$; (ii_2) if $(G,\tau) \in \mathcal{S}_{cg}$ then $G^\# \in \mathcal{S}_{cg}$.

Proof. (i_1) and (i_2) follow immediately from Theorem 5.1 (e). For (ii_1) and (ii_2), note that the identity map $G^\# \to (G,\tau)$ is continuous as $G^\#$ is the group G equipped with the finest precompact group topology. Now Lemma 3.1.1 applies. ☐

5.3. Lemma. *Let G be a direct sum of countable Abelian groups. Then $G^\# \in \mathcal{S}_g$.*

Proof. Let $\{G_i : i \in I\}$ be a family of countable subgroups of an Abelian group G such that $G = \oplus_{i\in I} G_i$. By Theorem 2.2 of [C–T], every $G_i^\#$ is in $\mathcal{S}_{cg}$. Then, by Corollary 3.2.4, the group $G = \oplus_{i \in I} G_i^\#$ equipped with the Tikhonov topology τ with respect to the summands $G_i^\#$ is in $\mathcal{S}_g$ as well. Since the topology τ is precompact, we can apply Proposition 5.2 to conclude that $G^\# \in \mathcal{S}_g$. ☐

We give now an immediate corollary to Lemma 5.3 though we shall prove later a stronger result (for (a) and (c) see Lemma 5.5, for (b) see Lemma 5.6).

5.4. Corollary. *$G^\# \in \mathcal{S}_g$ whenever an Abelian group G satisfies one of the following conditions:*

(a) G is algebraically free;

(b) G is divisible;

(c) G is a direct sum of cyclic groups.

Our positive results about $G^\#$ show that the lack of non-trivial convergent sequences need not always imply the lack of a suitable set (cf. Example 2.12). One of the reasons is that the group $G^\#$ is never countably compact for an infinite group G (by Theorem 5.1 (c)).

It is known that the group $\mathbf{J}_p$ of p-adic integers is indecomposable. Moreover, the compact group $\mathbf{J}_p$ is monothetic, and hence $\mathbf{J}_p \in \mathcal{F} \subseteq \mathcal{S}_c$. However, $\mathbf{J}_p \notin \mathcal{S}_g$

by Proposition 2.2 (c), so that Corollary 5.4 cannot be applied to decide whether $\mathbf{J}_p^\# \in \mathcal{S}$. However, we shall see below that $\mathbf{J}_p^\# \in \mathcal{S}_{cg}$.

Let us show that if G is a direct sum of cyclic Abelian groups then the stronger property $G^\# \in \mathcal{S}_{cg}$ holds.

5.5. Lemma. *$G^\# \in \mathcal{S}_{cg}$ holds when G is a direct sum of cyclic groups.*

Proof. By our hypothesis G is generated by an independent set S and by (f) of Theorem 5.1 the set S is closed and discrete. Thus, $G^\# \in \mathcal{S}_{cg}$. □

5.6. Lemma. *$G^\# \in \mathcal{S}_{cg}$ hold for every divisible Abelian group.*

Proof. Let G be a divisible abelian group. If $|I| \leq \aleph_0$, the conclusion follows from Theorem 2.2 of [C–T], so that we assume that $|I| > \aleph_0$. Then one can easily see from the structure theorem of divisible Abelian groups that $G \cong H \times D \times D$, where H is a countable Abelian group and D is an infinite Abelian group (cf. [Ro, 4.1.5], note that both H and D are also divisible as direct summands of G but this is irrelevant for our proof). By Theorem 2.2 of [C–T], $H^\#$ is in $\mathcal{S}_{cg}$. On the other hand, by Theorem 5.1 (g), the group $D^\#$ contains a closed discrete subset of size $|D|$. Now Lemma 2.4 (b) yields that $D^\# \times D^\# \in \mathcal{S}_{cg}$. By Proposition 3.2.5, $H^\# \times D^\# \times D^\# \in \mathcal{S}_{cg}$. It remains to apply Theorem 5.1 (d) to conclude that $G^\# \cong H^\# \times D^\# \times D^\# \in \mathcal{S}_{cg}$. □

5.7. Theorem. *$G^\# \in \mathcal{S}_{cg}$ for every Abelian group G.*

Proof. Let G be a torsion Abelian group. Then there exists a subgroup H of G which is a direct sum of cyclic subgroups such that the quotient group G/H is divisible (see Theorem 2.3.4 of [Ro]). Now Lemma 5.6 gives $(G/H)^\# \in \mathcal{S}_{cg}$, while Lemma 5.5 implies $H^\# \in \mathcal{S}_{cg}$. By Theorem 3.4.5 we conclude that $G^\# \in \mathcal{S}_c$. Hence $G^\# \in \mathcal{S}_{cg}$ by Proposition 5.2 (i_2).

Let G now be an *arbitrary* Abelian group. Fix a free subgroup F of G such that G/F is a torsion group. By the above argument, $(G/F)^\# \in \mathcal{S}_{cg}$, while $F^\# \in \mathcal{S}_{cg}$ by Lemma 5.5. Finally, Remark 3.4.6 implies that $G^\# \in \mathcal{S}_{cg}$. □

Let us observe that this theorem gives a class of groups in $\mathcal{S}_{cg}$ quite far from those in $\mathcal{S}_c$ provided by Theorem 0.3 (b) since now $d(G^\#) = |G|$ and $b(G^\#) = \omega$ by Theorem 5.1 (a), while Theorem 0.3 (b) applies to groups G with $d(G) < b(G)$.

The next theorem generalizes Theorem. 5.7. Let us recall that every locally compact Abelian group has the form $G = \mathbb{R}^n \times H$, where H has an open compact subgroup K. While the vector subgroup $\mathbb{R}^n$ is uniquely determined, the direct summand H is not. We shall refer to K as to *maximal compact subgroup* of G, since for every compact subgroup L of G the sum $K + L$ is compact and the index $[(K + L) : K]$ is finite.

5.8. Theorem. *$G^+ \in \mathcal{S}$ for every locally compact Abelian group G. Moreover, if G has a maximal compact subgroup that admits a closed suitable set, then also $G^+ \in \mathcal{S}_c$.*

Proof. Let $G = \mathbb{R}^n \times H$, where H has an open compact subgroup K. Consider the open subgroup $G_1 = \mathbb{R}^n \times K$ of G. Since $\mathbb{R}^n$ has a dense $n+1$ generated subgroup ([CM, Proposition 2.3]), we conclude that $(\mathbb{R}^n)^+ \in \mathcal{S}_c$. Now $G_1^+ = (\mathbb{R}^n)^+ \times K$ by (d), hence Corollary 3.2.4 yields $G_1^+ \in \mathcal{S}$ as $K \in \mathcal{S}$ (by Theorem 0.2). Moreover, by Theorem 3.1.2 $G_1^+ \in \mathcal{S}_c$ when $K \in \mathcal{S}_c$.

Observe that by Theorem 5.1 (d) the quotient G^+/G_1 has the same topology as $(G/G_1)^+$ and the induced from G^+ topology on G_1 coincides with that of G_1^+. Since the subgroup G_1 is open, $(G/G_1)^+ = (G/G_1)^\#$. Hence by Theorem 5.7, the quotient $G^+/G_1^+ = (G/G_1)^\#$ has a closed suitable set. Now by Theorem 3.4.5, applied to G^+ and G_1^+ we conclude that $G^+ \in \mathcal{S}$. The same theorem permits to conclude that $G^+ \in \mathcal{S}$ whenever $K \in \mathcal{S}_c$ since this was shown above to yield $G_1^\# \in \mathcal{S}_c$. ☐

It is easy to see that if some maximal compact subgroup of a locally compact Abelian group G has a closed suitable set, then all maximal closed subgroups of G have that property. In particular, this applies to discrete Abelian groups, hence Theorem 5.8 is indeed a strengthening of Theorem 5.7.

5.9. Corollary. *Let G be a direct product of locally compact Abelian groups. Then G^+ has a suitable set.*

Proof. Follows from Theorem 5.1 (d) and Theorem 5.8. ☐

We are left with the following questions:

5.10. Question. *(a) Let G be a locally compact Abelian group. Does $G \in \mathcal{S}_c$ always imply $G^+ \in \mathcal{S}_c$?*

(b) Does every topological Abelian group that satisfy Pontryagin duality admit a suitable set?

The motivation for (a) is the general fact that many properties of G are preserved under the passage to G^+ (see Theorem 5.1, (b), (c)). Having a suitable set is such an example, according to Theorems 0.2 and 5.8. A positive evidence supporting the counterpart for closed suitable set (i.e., question (a)) is the second part of Theorem 5.8 and the fact that a locally compact Abelian group G as described there admits a closed suitable set (although there are many locally compact Abelian groups $G \in \mathcal{S}_c$ that do not satisfy that property, according to Proposition 2.8). As a last fact supporting (a) we mention that we do not see any examples of locally compact Abelian groups G violating $G^+ \in \mathcal{S}_c$ except in the extreme case $G = G^+$ compact.

Part (b) of Question 5.10 is motivated by Corollary 5.9 since direct products of locally compact Abelian groups satisfy Pontryagin duality ([Ka]). Closed subgroups of countable products of locally compact Abelian groups satisfy Pontryagin duality as well [N]. A wealth of material upon which this question could be tested is provided by the important monograph [B].

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Note added February 2, 1998: The first part of Theorem 5.8 was proved independently also by Tomita and Trigos [TT]. Recently Trigos and the first named author answered Question 5.10: part (a) positively and part (b) negatively. The proofs will appear in a forthcoming paper.

REFERENCES

[B] W. Banaczcyk, *Additive Subgroups of Topological Vector Spaces*, Lecture Notes in Mathematics **1466**, Springer-Verlag, Berlin, 1991.

[CM] J. Cleary, S.A. Morris, Generators for locally compact groups, *Proc. Edinburgh Math. Soc.* **36** (1993), 463–467.

[CMRST] W.W. Comfort, S.A. Morris, D. Robbie, S. Svetlichny and M. Tkačenko, Suitable sets for topological groups, *Topology Appl.*, **86**, no. 1 (1998).

[CRn] W.W. Comfort and L. Robertson, Extremal phenomena in certain classes of totally bounded groups, *Dissertationes Mathematicae* **272** (1988), 48 pp.

[CRs] W.W. Comfort and K.A. Ross, Pseudocompactness and uniform continuity in topological groups, *Pacific J. Math.* **16** (1966), 483–496.

[CSa] W.W. Comfort and V. Saks, Countably compact groups and finest totally bounded uniformities, *Pacific J. Math.* **49** (1973), 33–44.

[DiSw] S. Dierolf and U. Schwanengel, Un example d'un groupe topologique q-minimal mais non précompact, *Bull. Sci. Math.* (2) **101** (1977), 265–269.

[D1] D. Dikranjan, Zero-dimensionality of some pseudocompact groups, *Proc. Amer. Math. Soc.* **120** (1994), 1299–1308.

[D2] D. Dikranjan, Recent advances in minimal topological groups, *VIII Prague Symposium on Topology and its Relation to Modern Algebra and Analysis* (Prague, August 1996), Topology Appl., submitted.

[D3] D. Dikranjan, Structure of the countably compact minimal groups, Preprint.

[DP] D. Dikranjan and Iv. Prodanov Totally minimal topological groups, *Annuaire Univ. Sofia Fac. Math. Méc.* **69** (1974/75), 5-11.

[DPS] D. Dikranjan, Iv. Prodanov and L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, Pure and Applied Mathematics, Vol. **130**, Marcel Dekker Inc., New York-Basel, 1989, pp. 287+x.

[DS] D. Dikranjan and D. Shakhmatov, Compact-like totally dense subgroups of compact groups, *Proc. Amer. Math. Soc.* **114**, (1992), 1119-1129.

[DTT] D. Dikranjan, M.G. Tkačenko and V.V. Tkachuk, Some topological groups with and some without suitable set, *Topology Appl.*, to appear.

[Do] A. Douady, *Cohomologie des groupes compacts totalements discontinus*, Séminaire Bourbaki 1959/60, Exposé 189, Secrétariat Math. Paris, 1960.

[vD1] E.K. van Douwen, The product of two countably compact topological groups, *Trans. Amer. Math. Soc.* **262** (1980), 417–427.

[vD2] E.K. van Douwen, The maximal totally bounded group topology on G and the biggest minimal G-space for Abelian groups G, *Topology Appl.* **34** (1990), 69–91.

[Gl] I. Glicksberg, Uniform boundedness for groups, *Canad. J. Math.* **14** (1962), 269–276.

[Gr] M.I. Graev, Free topological groups, *Amer. Math. Translat.* **8** (1962), 325–361.

[Gu] I. Guran, Topological groups close to being Lindelöf, *Soviet Math. Dokl.* **23** (1981), 173–175.

[HJ] A. Hajnal and I. Juhász, A separable normal topological group need not be Lindelöf, *Gen. Topology Appl.* **6** (1976), 199–205.

[HvM2] K. Hart and J. van Mill, Discrete sets and the maximal totally bounded group topology, *J. Pure Appl. Algebra* **70** (1991), 73–80.

[HR] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, Berlin 1963.

[HM] K.-H. Hofmann and S.A. Morris, Weight and $\mathfrak{c}$, *J. Pure Applied Algebra* **68** (1990), 181–194.

[HdV] M. Hušek and J. de Vries, Preservation of products by functors close to reflectors, *Topology Appl.*, **27** (1987), 171-189.

[Ju] I. Juhász, On closed discrete subspaces of product spaces, *Bull. Acad. Polon. Sci. Ser. Math.* **17** (1969), 219-223.

[Ka] S. Kaplan, Extensions of Pontryagin duality I, Infinite products, *Duke Math. J.* **15** (1948), 649–658.

[K] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, Studies in Logic **102**, North-Holland, Amsterdam, 1980.

[M] O. Mel'nikov, Normal subgroups of free profinite groups, *Math. USSR-Izv.* **12** (1978), 1-20.

[Mo] S.A. Morris, Varieties of topological groups, *Bull. Austral. Math. Soc.* **1** (1969), 145-160.

[N] N. Noble, k-groups and duality, *Trans. Amer. Math. Soc.* **151** (1970), 551–561.

[P] Iv. Prodanov, Some minimal group topologies are precompact, *Math. Ann.* **227** (1977), 117–125.

[Ro] D.J.S. Robinson, *A course in the Theory of Groups,* Graduate Texts in Math. **80**, Springer Verlag New York Inc., 1982.

[Tk] M.G. Tkačenko, Countably compact and pseudocompact topologies on free Abelian groups, *Soviet Math. (Izv. VUZ)* **34** (1990), 79–86.

[TT] A. Tomita and F.J. Trigos-Arrieta, Suitable sets in products of topological groups and in groups equipped with the Bohr topology, in: D. Dikranjan and L. Salce, eds., *Abelian Groups, Module Theory and Topology: Proceedings in Honour of Adalberto Orsatti's 60-th birthday,* Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York-Basel, to appear.

[Tr] F.J. Trigos-Arrieta, Continuity, boundedness, connectedness and the Lindelöff property, *J. Pure Appl. Algebra* **70** (1991), 199–210.

[V] N. Varopoulos, Studies in harmonic analysis, *Proc. Camb. Phil. Soc.* **60** (1964), 465–516.

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